

## Written exam

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### Exercise 1

Consider the UNIVERSITY HIRING decision problem:

A university needs to hire the teaching staff for a new degree, for which a set  $T$  of topics must be taught. The executive board received  $n$  applications from prospective teachers, and every applicant  $i \in \{1, \dots, n\}$  has knowledge of a subset  $S_i \subseteq T$  of the required topics. The budget allows for hiring at most  $k \leq n$  teachers. Is there a choice of  $k$  applicants so that all teaching topics are covered?

An instance of the problem consists of parameters  $n, k, T, S_1, \dots, S_n$ .

**1.1)** Prove that UNIVERSITY HIRING  $\in \mathbf{NP}$ .

**1.2)** Prove by reduction that UNIVERSITY HIRING is complete in the class  $\mathbf{NP}$ . The reduction can refer to any language discussed during the course.

**1.3)** Prove that if  $k$  is kept constant (e.g.,  $k = 10$ ), then the problem's asymptotic complexity is polynomial wrt input size.

### Exercise 2

**2.1)** When is a language recursive? When is it recursively enumerable?

**2.2)** Prove that the following property  $\mathcal{P}$  of Turing machines  $\mathcal{M}$  is not recursive:

$$\mathcal{P} = \{\mathcal{M} : \mathcal{M}(\varepsilon) \text{ halts after an even number of steps}\}$$

where  $\varepsilon$  is the empty input string.

**2.3)** Prove that  $\mathcal{P}$  is recursively enumerable.

Hint — *Point 2.3 can be proved by explicitly outlining an enumeration algorithm.*

### Exercise 3

Consider the following language:

$$S = \left\{ (x \in \{0, 1\}^*, k \in \mathbb{N}) : x \text{ contains a subsequence of } k \text{ adjacent 0's} \right\}.$$

For example,  $(101000101, 3) \in S$  because the binary string contains 3 consecutive zeroes, while  $(101000101, 4) \notin S$  because the binary string does not contain 4 consecutive zeroes.

**3.1)** Prove that  $S \in \mathbf{P}$ .

**3.2)** Prove that  $S \in \mathbf{L}$ .

Hint — *Again, both points can be proved by describing an algorithm and showing that it has the required property.*